Critical Branching Random Walk conditioned to survive at a given set in $\mathbb{Z}^2$

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- At time $n+1$, any particle u of the n-th generation dies and produces independently N_u children and from S_u, the position of u, each child makes a jump according to X. N_u is distributed as $\{p_k; k \ge 0\}$.
- For any $B \subset \mathbb{Z}^d$,

$$Z_n(B) := \sum_{|u|=n} 1_{S_u \in B},$$

where $|u|$ denotes the generation of u. $\{Z_n = Z_n(\mathbb{Z}^d)\}_{n\ge 0}$ is a critical GW process which becomes extinct a.s.

Survival probability in B at time n

For any fixed finite set $B \subset \mathbb{Z}^d$,

$$\mathbb{P}_0(Z_n(B) \geq 1) \sim ?$$

- Kolmogorov'1938 showed that $\mathbb{P}(Z_n \geq 1) \sim \frac{2}{\sigma^2 n}$ as $n \to \infty$.

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- When $d \geq 3$, Rapenne'2022+ obtains that

$$\mathbb{P}_0(Z_n(B) \geq 1) \sim Constant \times P_n(B) \sim \frac{C_0|B|}{n^{d/2}}.$$

where $P_n(B) = \mathbb{P}_0(S_n \in B)$ with $\{S_n\}_{n\geq 0}$ simple random walk in $\mathbb{Z}^d$. Moreover,

$$\mathbb{P}_0(Z_n(B) \in \cdot | Z_n(B) \geq 1) \to p_B(\cdot).$$

How about $d = 2$?

- Durrett'1979 proved for critical branching Brownian motion in $\mathbb{R}^2$, if B is a bounded open set with $|\partial B| = 0$,

$$\mathbb{P}_0\left(\frac{8\pi}{\log n}\frac{Z_n(B)}{|B|} > h\right) \sim e^{-h}\frac{4}{n\log n}, \forall h > 0.$$

$$\Rightarrow \mathbb{P}_0(Z_n(B) \geq 1) = \mathbb{P}_0\left(\frac{8\pi}{\log n}\frac{Z_n(B)}{|B|} > 0\right) \gtrsim \frac{4}{n\log n}.$$

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- Lalley-Zheng'2011 proved that if $B = \{x\} \subset \mathbb{Z}^2$, offspring is critical binary $p_0 = p_2 = 1/2$, motion is SRW,

$$\frac{P_n(x)}{C_3 + C_4 \log n} \leq \mathbb{P}_0(Z_n(x) \geq 1) \leq \frac{C_1}{n\log n}\exp\left(-C_2\frac{|x|^2}{n}\right)$$

where $P_n(x) = \mathbb{P}_0(S_n = x)$ with $\{S_n\}_{n\geq 0}$ simple random walk. Note that

$$P_n(x) = \frac{5}{4\pi n}e^{-\frac{5|x|^2}{4n}} + \frac{o_n(1)}{n}.$$

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Yaglom theorem for CBRW [C. He and Lin'2022+, in progress]

When $d = 2$, if $\sum_k e^{\delta k} p_k < \infty$ for some $\delta > 0$, then

1. uniformly for $x \in \mathbb{Z}^2$,
$$\mathbb{P}_0(Z_n(x) \geq 1) = \frac{P_n(x)}{c_{SRW}\sigma^2 \log n} + o(\frac{1}{n \log n});$$
where $P_n(x) \sim \frac{c_{SRW}}{n}$ if $|x| = o(\sqrt{n})$.
2. for any fixed bounded set $B \subset \mathbb{Z}^2$,
$$\mathbb{P}_0(Z_n(B) \geq 1) \sim \frac{4}{\sigma^2 n \log n};$$
3. further, for $z_n \in \mathbb{Z}^2$ such that $z_n/\sqrt{n} \to z \in \mathbb{R}^2$,
$$\mathcal{L}(\frac{Z_n(z_n + B)}{c_{SRW}|B| \log n} | Z_n(z_n + B) \geq 1) \to Exp(1).$$

Critical BRW in $\mathbb{Z}^2$

classical Yaglom theorem

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Theorem[C. He and Lin'2022+, in progress]

When $d = 2$, if $\sum_k e^{\delta k} p_k < \infty$ for some $\delta > 0$, then for any $|x| = O(\sqrt{n})$,

$$\mathcal{L}(\frac{Z_n}{\sigma^2 n/2}, \frac{Z_n(x)}{c_{SRW} \log n}|Z_n(x) \geq 1) \to (\underbrace{\Gamma(2,1), Exp(1)}_{indep.}).$$

CBRW at typical position

At time n, given $\{Z_n \geq 1\}$, choose uniformly one particle among Z_n alive ones and denote its position by S_n^* which is called typical position.

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Let $\Omega_n = \sum_{x \in \mathbb{Z}^2} \mathbf{1}_{\{Z_n(x) \geq 1\}}$ be the number of occupied sites.
Lalley-Zheng'2011 showed that conditioned on $\{Z_n \geq 1\}$,

$$\Omega_n = O_{\mathbb{P}}\left(\frac{n}{\log n}\right), \;\; V_n = \max_x Z_n(x) = O_{\mathbb{P}}(\log n)^2.$$

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Survival at two sites

Observations:

- For Ω_n the total number of occupied sites at time n,

$$\mathbb{E}_0[\Omega_n] = \sum_{x \in \mathbb{Z}^2} \mathbb{P}_0(Z_n(x) \geq 1) \approx \sum_x \frac{P_n(x)}{c_{SRW}\sigma^2 \log n} = \frac{1}{c_{SRW}\sigma^2 \log n}.$$

So, $\mathbb{E}_0[\Omega_n | Z_n \geq 1] \sim c_\Omega \frac{n}{\log n}$. The second moment is

$$\mathbb{E}_0[\Omega_n^2] = \sum_{x,z} \mathbb{P}_0(Z_n(x) \geq 1, Z_n(z) \geq 1)$$

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Question: $\mathbb{P}_0(Z_n(x) \geq 1, Z_n(x+z_n) \geq 1)$ with $|x| = O(\sqrt{n})$ and $|z_n| = \ell_n \in [1, \sqrt{n}]$.

Theorem[C. He and Lin'2022+, in progress]

When $d = 2$, if $\sum_k e^{\delta k} p_k < \infty$ for some $\delta > 0$, then for $\ell_n = |z_n|$ and $|x| = O(\sqrt{n})$,

1. if $\ell_n = n^{o(1)}$, then
$$\mathbb{P}_0(Z_n(x) \geq 1, Z_n(x + z_n) \geq 1) \sim \mathbb{P}_0(Z_n(x) \geq 1);$$
2. if $\ell_n = n^a$ with $a \in (0, 1/2)$, then
$$\mathbb{P}_0(Z_n(x) \geq 1, Z_n(x + z_n) \geq 1) \sim \mathbb{P}_0(Z_n(x) \geq 1)\frac{1-2a}{1-a};$$
3. if $z_n/\sqrt{n} \to z_1 \neq 0$ and $x/\sqrt{n} \to z_0$ ($\ell_n \sim constant \times \sqrt{n}$), then
$$\mathbb{P}_0(Z_n(x) \geq 1, Z_n(x + z_n) \geq 1) \sim \mathbb{P}_0(Z_n(x) \geq 1)\frac{\gamma(z_0, z_1)\sigma^2}{\log n}.$$

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1. if $\ell_n = n^{o(1)}$, then

$$\mathcal{L}\left(\frac{Z_n(x+z_n)}{c_{SRW}\log n}, \frac{Z_n(x)}{c_{SRW}\log n} \Big| Z_n(x) \geq 1\right) \to (Y, Y).$$

   where Y is Exp(1) random varaible.

2. if $z_n/\sqrt{n} \to z_1 \neq 0$ and $x/\sqrt{n} \to z_0$ ($\ell_n \sim constant \times \sqrt{n}$), then

$$\mathcal{L}\left(\frac{Z_n(x+z_n)}{c_{SRW}\log n}, \frac{Z_n(x)}{c_{SRW}\log n} \Big| Z_n(x) \geq 1, Z_n(x+z_n) \geq 1\right) \to (Y', Y)$$

   where Y', Y are independent Exp(1) random variables.

Conjecture on survival at two far away sites

If $|z_n| = n^a$ with $a \in (0, 1/2)$, we conjecture that

$$\mathcal{L}\left(\frac{Z_n(x+z_n)}{c_{SRW}\log n}, \frac{Z_n(x)}{c_{SRW}\log n} \Big| Z_n(x) \geq 1, Z_n(x+z_n) \geq 1\right) \to (Y_a, Y)$$

where Y_a and Y are correlated.

Moreover, given $\{Z_n(x) \geq 1\} \cap \{Z_n(z) \geq 1\}$, one could consider the most recent common ancestor of particles at $\{x, z\}$.

Number of occupied sites Ω_n

Consequently,

$$\mathbb{E}_0[\Omega_n^2 | Z_n \geq 1] \sim 2c_\Omega^2 n^2/(\log n)^2.$$

Recall that $\mathbb{E}_0[\Omega_n | Z_n \geq 1] \sim c_\Omega \frac{n}{\log n}$.

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Theorem[C. He and Lin'2022+, in progress]

When $d = 2$, if $\sum_k e^{\delta k} p_k < \infty$ for some $\delta > 0$, then

$$\mathcal{L}(\frac{\Omega_n}{c_\Omega n/\log n} | Z_n \geq 1) \to Exp(1).$$

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$$\mathcal{L}\left(\frac{\Omega_n}{c_\Omega n/\log n} | Z_n \geq 1\right) \to Exp(1).$$

Lalley-Zheng'2011 proved that when $d \geq 3$,

$$\mathcal{L}\left(\frac{\Omega_n}{cn}, \frac{Z_n}{\sigma^2 n/2} | Z_n \geq 1\right) \to (Y, Y).$$

with $Y \sim Exp(1)$.

1. Take the most recent common ancestor a_n of all alive particles in Z_n, let its generation be U_n. Note that a_n has 2 children which have descendants in Z_n with probability $1 - o_n(1)$. And $U_n/n \Rightarrow U[0,1]$.

$$\frac{\Omega_n}{n/\log n} = \frac{\Omega^{(1)}_{n-U_n} + \Omega^{(2)}_{n-U_n} - \text{ intersection of 2 sub-families}}{n/\log n}$$

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2. Intersection part is negligible. So the limiting dist. equation is

$$\Omega \overset{d}{=} U(\Omega^{(1)} + \Omega^{(2)})$$

where $U \sim U[0,1]$ independent of others, $\Omega^{(1)}$, $\Omega^{(2)}$ are i.i.d. copies of Ω. So, Ω is of exponential dist.

3. Mallows distance on probability measures can be applied here

$$d(\mu,\nu) = \inf_{X\sim\mu, Y\sim\nu} \sqrt{\mathbb{E}[(X-Y)^2]}$$

Conjectures

1. $V_n = \max_x Z_n(x)$? $V_n = \Theta_{\mathbb{P}}(\log n)$ when $d \geq 3$. Similarly,

$$\frac{V_n}{\log n} \approx \max\{\frac{V^{(1)}_{n-U_n}}{\log n}, \frac{V^{(2)}_{n-U_n}}{\log n}\} \Longrightarrow V \stackrel{d}{=} \max\{V^{(1)}, V^{(2)}\}$$

   So V is some constant $c \in [0, \infty]$. Conjecture: $V_n / \log n \xrightarrow{\mathbb{P}} c_d \in (0, \infty)$ for $d \geq 3$.

2. We should have $V_n = \Theta_{\mathbb{P}}(\log n)^2$ when $d = 2$. Conjecture: $V_n / (\log n)^2 \xrightarrow{\mathbb{P}} c_2$ for $d = 2$.
3. In stead of Mallows distance, what kind of distance can we use here?

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